

Improving Ranking Using Quantum Probability

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Abstract

Data management systems, like database, information extraction, information retrieval or learning systems, store, organize, index, retrieve and rank information units, such as tuples, objects, documents, items to match a pattern (e.g. classes and profiles) or meet a requirement (e.g., relevance, usefulness and utility). To this end, these systems rank information units by probability to decide whether an information unit matches a pattern or meets a requirement. Classical probability theory represents events as sets and probability as set measures. Thus, distributive and total probability laws are admitted. Quantum probability is a non-classical theory nor does admit distributive and total probability laws. Although ranking by probability is far from being perfect, it is optimal thanks to statistical decision theory and parameter tuning.

The main question asked in the paper is whether further improvement over the optimality provided by probability may be obtained if the classical probability theory is replaced by quantum probability theory. Whereas classical probability (and detection theory) is based on sets such that the regions of acceptance / rejection are set-based detectors, quantum probability is based on subspace-based detectors.

The paper shows that ranking information units by quantum probability differs from ranking them by classical probability provided the same data used for parameter estimation. As probability of detection (also known as recall or power) and probability of false alarm (also known as fallout or size) measure the quality of ranking, we point out and show that ranking by quantum probability yields higher probability of detection than ranking by classical probability provided a given probability of false alarm and the same parameter estimation data.

As quantum probability provided more effective detectors than classical probability within other domains than data management, we conjecture that, the system that can implement subspace-based detectors shall be more effective than a system which implements a set-based detectors, the effectiveness being calculated as expected recall estimated over the probability of detection and expected fallout estimated over the probability of false alarm.

1 Introduction

Data management systems, like database, information retrieval (IR), information extraction (IE) or learning systems, store, organize, index, retrieve and rank information units, like tuples, objects, documents, items. A wide range of applications of these systems have emerged that require the management of uncertain or imprecise data. Important examples of data are sensor data, webpages, newswires, imprecise attribute values. What is common to all these applications is uncertainty and then that they have to deal with decision and statistical inference.

Ranking is perhaps the most crucial task performed by the data management systems which have to deal with uncertainty. In many applications, ranking aims at deciding or inferring, for example, the class assigned to a unit or the order by relevance, usefulness, or utility of the units delivered to another application or to an end user. In addition, ranking is performed to decide whether a unit is placed at a given rank.

The management of imprecise data require means for ranking information units by probability. Ranking places information units in a list ordered by a measure of utility, cost, relevance, etc.. A probability theory measures the uncertainty of the decision. To this end, the definition of an event space and the estimation of probabilities are necessary steps for representing imprecise data and making predictions within many contexts of data management like machine learning, information retrieval or probabilistic databases.

The measurement of the imprecision and the uncertainty in the data leads to the definition of regions of acceptance of a predefined set of hypotheses, thus bringing many decision problems to the calculation of a probability of detection and of a probability of false alarm. Although the data management systems reach good results thanks to classical probability theory and parameter tuning, ranking is far from being perfect because useless units are often ranked at the top or useful units are missed.

Classical probability theory describes events and probability distributions using sets and set measures, respectively, according to Kolmogorov's axioms [13]. In contrast, quantum probability theory describes events and probability distributions using Hermitian operators in the complex Hilbert vector space. Whereas parameter tuning is performed within a fixed probability theory, the adoption of quantum probability entails a radical change. Furthermore, whereas classical probability is based on sets such that the regions of acceptance or rejection are set-based detectors (i.e., indicator func-

tions), quantum probability is based on subspace-based detectors and the detectors are projector-based. Note that the use of quantum probability does not imply that quantum phenomena are investigated in the paper; we are interested in the formalism based the Hilbert vector spaces instead.

The main question asked in the paper is whether further improvement may be obtained if the classical probability theory is replaced by the quantum probability theory. The paper shows that ranking information units by quantum probability yields different outcomes which are in principle more effective than ranking them by classical probability given the same data available for parameter estimation. The effectiveness is measured in terms of probability of detection (also known as recall or power) and probability of false alarm (also known as fallout or size).

We structure the paper as follows. Section 2 illustrates the basics of the probability theory through a view that encompasses both theories. Section 3 compares quantum detection with classical detection. Section 4 shows that the ranking by quantum probability more effective than the ranking by classical probability. Section 5 provides an interpretation of the projectors which define the regions of acceptance and rejection. Section 6 describes the algorithm for ranking information units by quantum probability. Section 7 provides an overview of the related work.

2 Classical Probability and Quantum Probability

In this section, we introduce a special view of probability distributions for the classical theory of probability. The same view is also introduced for quantum probability, which is a non-classical theory and does not admit the distributive law, to provide a general framework for quantum and classical probabilities; the view is depicted in Figure 1.

Before introducing the view of probability theory, some basic definitions are provided. A probability space is a set of mutually exclusive events such that each event is assigned a probability between 0 and 1 and the sum of the probabilities over the set of events is 1. For the sake of clarity, we introduce the case of binary event spaces because it is the simplest and most common in data management – keyword occurrence in webpages, binary features in sample records or binary attribute values in relational tables are

some examples. The case of binary event spaces are usually represented by mutually exclusive scalars like 0 and 1. If binary scalars are used, the mutual exclusiveness is given by the scalar product, for example, $01 = 0$ (see [4]).

Whereas the scalars $\{0, 1\}$ is a possible representation of events, the vectors of a complex finite-dimensional are another option. When using vectors, an event is $(0, 1)'$ and its complement is $(1, 0)'$. The representation of the events must encode the mutual exclusiveness. If binary vectors are used, the mutual exclusiveness is given by the inner product, i.e., $(0, 1)(1, 0)' = 0$.

When the event space is not binary (e.g., when the events are represented by k natural numbers $0, 1, \dots, k-1$), a binary representation can again be used. The vector $(0, \dots, 0, 1)'$ is assigned to symbol 0, the vector $(0, \dots, 1, 0)'$ is assigned to symbol 1, and so on until $(1, \dots, 0, 0)'$ is assigned to symbol $k-1$. Whatever the representation is used, the inner product between two vectors must be 0 and their norm must be 1.

The mapping between the probabilities and the events is called “probability distribution” which is a function mapping a mathematical object which represents an event to a real number ranging between 0 and 1. The difference between classical probability and quantum probability; the difference is due to the way the event space and the probability distribution are represented.

The starting point of the view of probability used in the paper is the algebraic form of the probability space. To this end, Hermitian¹ (or self-adjoint) linear operators are used. In quantum mechanics, “operator” is preferred to “matrix” yet in the paper, for the sake of clarity, “matrix” is preferred because, for a fixed basis, the matrices are isomorphic to the operators. A matrix is Hermitian when it is equal to its conjugate transpose. Hermitian matrices are important because their eigenvalues are always real. In particular, Hermitian matrices with trace 1 is the key notion in quantum probability because the sum of the eigenvalues is 1 and, thus, the eigenvalues can be viewed as a probability distribution.

The projector is an idempotent Hermitian matrix. Every subspace has one projector and then the projectors are 1:1 correspondence with the subspaces. Each vector corresponds to one projector with rank one defined as the outer product of the vector by its conjugate transpose. There are two main instructions for representing events using projectors:

- the projectors must be mutually orthogonal for representing the mutual exclusiveness of the events, and

¹“Symmetric” is adopted in the real field.

- the projectors must have trace 1 for making probability calculation consistent with the probability axioms.

An event space and a probability function defined over it are represented using Hermitian matrices with trace 1. In particular, a projector represents an event and an event space is modeled by a collection of projectors. As the union of the events results in the whole event space, the sum of the projectors of a collection corresponding to an event space results in the unity. More specifically, if $\{\mathbf{E}_0, \dots, \mathbf{E}_{k-1}\}$ is a collection of mutually orthogonal projectors,

$$\mathbf{E}_0 + \dots + \mathbf{E}_{k-1} = \mathbf{I} ,$$

the latter being termed “resolution to the unity”. For example, using the Dirac notation introduced in Appendix A, the projector of two events are represented by

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1)$$

and

$$|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2)$$

However, there is not a unique representation of an event space. For example, the following vectors are also representing mutually exclusive events:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (3)$$

thus leading to a different resolution to the unity given by the following projectors

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (4)$$

The second kind of Hermitian matrix of a probability space is the density matrix; the density matrix encapsulates the probability values assigned to the events. In Physics, a density matrix represents the *state* of a microscopic system, such as a particle, a photon, etc.. The structure of a microscopic system is unknown. Yet a device can measure the system to obtain some information. A microscopic system is similar to a urn of colored balls. The

internal composition of the urn is always unknown even if opened and observed because the device disturbs the state (i.e., the distribution of the colors) of the urn.

In data management and in other domains different from Particle Physics, a system is macroscopic instead.² Examples of macroscopic systems in data management are webpages, customers, queries, clicks, tuples, attributes, and so on. The states of these systems correspond to the probability densities according to which keywords, reviews, attribute values are observables to be measured from such systems. Density matrices are a powerful formalism in the macroscopic worlds too because they allow us to introduce the algebraic approach adopted for incorporating the more powerful probability space and decision rule suggested in the paper.

To the end of introducing the way density matrices are defined, consider two equiprobable events, e.g., the occurrence of a feature or a positive/negative customer review. The probability distribution is $(\frac{1}{2}, \frac{1}{2})$ where each value refers to an event. As an alternative to a list, the probability distribution can be arranged along the diagonal of a two-dimensional matrix and the other matrix elements are zeros. For example, the matrix corresponding to the probability distribution of two equally probable events is

$$\mu = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

In general, the probability distribution (p_1, \dots, p_k) of a k -event space can be written as

$$\mu = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k \end{pmatrix}$$

A probability distribution is *pure* when the density matrix is a projector, otherwise, the distribution is *mixed*. A distribution is mixed when the density matrix is a mixture of density matrices; a pure distribution is an instance of mixture with one matrix. The density matrix representing a pure distribution is 1:1 correspondence with a density vector such that the projector is the

²A macroscopic system in data management is thus not a computer systems as far as it is concerned in the paper.

outer product between the vector and its conjugate transpose. A classical probability distribution is pure when the probability is concentrated on a single elementary event which is the certain event and then has probability 1.

Given a density matrix, the spectral theorem helps find the underlying events and the related probabilities. Because of the importance of the spectral theorem, we provide its definition below:

Theorem 1 *To every Hermitian matrix \mathbf{A} on a finite-dimensional complex inner product space there correspond real numbers $\alpha_0, \dots, \alpha_{r-1}$ and rank-one projectors $\mathbf{E}_0, \dots, \mathbf{E}_{r-1}$ so that the α_j 's are pairwise distinct, the \mathbf{E}_j are mutually orthogonal, $\sum_{j=0}^{r-1} \mathbf{E}_j = \mathbf{I}$, $\sum_{j=0}^{r-1} \alpha_j = 1$ and $\sum_{j=0}^{r-1} \alpha_j \mathbf{E}_j = \mathbf{A}$.*

Proof See [10, page 156]. ■

The eigenvalues are the spectrum and $\mathbf{E}_0, \dots, \mathbf{E}_{r-1}$ are the projectors of the spectrum of \mathbf{A} . From Theorem 1, thus, a pure distribution is always a rank-one projector.

The spectral theorem says that any Hermitian matrix corresponding to a distribution can be decomposed as a linear combination of projectors (i.e. pure distributions) where the eigenvalues are the probability values associated to the events represented by the projectors. The eigenvalues are real because the decomposed matrix is Hermitian, are non-negative and sum to 1. For example, when the matrix corresponding to the distribution of two equally probable events is considered, the spectral theorem says that

$$\mu = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A mixed distribution have more non-zero eigenvalues, a pure distribution has a single eigenvalue 1.

In classical probability, every pure distribution represented by a diagonal density matrix corresponds to a projector. However, in general, a density matrix is not necessarily diagonal, yet the matrix is necessarily Hermitian. For example, (4) are trace-one projectors and correspond to pure distributions, thus there is a certain event (with probability 1) and an impossible event (with probability 0, of course). Yet, they are not diagonal. When, for example, keyword occurrence in webpages is represented, the first projector may be assigned eigenvalue 1 and the other is assigned eigenvalue 0. Thus the

former represents the certain event and the latter represents the impossible event in the probability space.

When using the algebraic form to represent probability spaces, the function for computing a probability is the trace of the matrix obtained by multiplying the density matrix by the projector corresponding to the event. The usual notation for the probability of the event represented by projector \mathbf{E} when the distribution is represented by density matrix ρ is

$$\text{tr}(\rho\mathbf{E}) \tag{5}$$

also known as Born's rule [23]. For example, when $\rho = \mu$

$$\mu = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the probability is

$$\text{tr}(\mu\mathbf{E}) = \text{tr} \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \right) = \frac{1}{2}$$

When $\mathbf{E} = |x\rangle\langle x|$ is a rank-one projector, the trace-based probability function can be written as

$$\text{tr}(\rho\mathbf{E}) = \langle x|\rho|x\rangle$$

When ρ is a rank-one projector $|y\rangle\langle y|$, then

$$\text{tr}(\rho\mathbf{E}) = |\langle x|y\rangle|^2$$

From the example, the definition of a function that computes the probability of an event when the probability is already allocated in the diagonal of the density matrix may be odd. However, we have shown that not all the density matrices corresponding to a distribution need to be diagonal matrices and the diagonal elements do not necessarily correspond to probability values, although they do have to sum to 1.

A density matrix encapsulate the values assigned to the events by a probability function because of Gleason's theorem stated below and proved in [12].

Theorem 2 *To every probability distribution on the set of all projectors in a complex vector space with dimension greater than 2 there corresponds a*

unique density matrix ρ on the same vector space for which the probability of the event represented by a projector $|x\rangle\langle x|$ is $\text{tr}(\rho|x\rangle\langle x|)$ for every unit vector x in the vector space.

Basically, the theorem tells us that corresponding to a probability distribution is one density matrix such that the probability of any event represented as a projector is calculated by the trace function.

The probability of an event when computed using a mixture differs from the probability computed using a pure state, yet they share the classical probability term whereas the difference is called *interference term*. Using a mixture,

$$\text{tr}(\mu|x\rangle\langle x|) = |a_0|^2|b_0|^2 + |a_1|^2|b_1|^2 \quad \mu = \begin{pmatrix} |a_1|^2 & 0 \\ 0 & |a_0|^2 \end{pmatrix} \quad (6)$$

Using superposition,

$$\text{tr}(\rho|x\rangle\langle x|) = |a_0|^2|b_0|^2 + |a_1|^2|b_1|^2 + 2|a_0||b_0||b_1||b_0| \cos \theta \quad (7)$$

where $\rho = |v\rangle\langle v|$ and θ is the angle of the polar representation of the complex number $a_0\bar{b}_0a_1\bar{b}_1$. Suppose, as an example, that $|x\rangle$ represents the event “the keyword occurs” and the density matrix represents the probability distribution of keyword occurrence in useful webpages. The common factor (i.e. (6)) is the sum of two probabilities; the probability that the webpage is not useful ($|a_0|^2$) multiplied by the probability that the keyword occurs in a useless webpage ($|b_0|^2$), and the probability that the webpage is useful ($|a_1|^2$) multiplied by the probability that the keyword occurs in a useful webpage ($|b_1|^2$). The sum is nothing but an application of the law of total probability.

The quantity $2|a_0||b_0||b_1||b_0| \cos \theta$ is the interference term. As the interference term ranges between -1 and $+1$, the probability of keyword occurrence computed when usefulness is superposed with uselessness becomes different from the common factor in which usefulness and uselessness are mutually exclusive and their probability distribution is described by a mixture. The interference term can be so large that the law of total probability is violated and any probability space obeying Kolmogorov’s axioms cannot admit the probability values $|a_1|^2$ and $|b_1|^2$, thus requiring the adoption of a quantum probability space [2, 3].

3 Quantum Probability and Decision

In general, the information stored in the data is acquired and delivered through information unit representation and ranking, these processes are described in terms of decision and estimation, and they are therefore affected by error. The error could be eliminated only if precise and exhaustive methodological tools and computer systems were developed. Nevertheless, there is a trade-off between precision, exhaustivity and the computation cost because high level of the former can be achieved only if a high computation cost is devoted. Thus, a certain amount of error is unavoidable yet can be controlled and limited below a given threshold.

Either a set of statements, or hypotheses, must be decided to best describe the information unit insofar as data permit to judge (e.g., the best topic(s) to which a webpage is assigned), or the values of certain quantities (also known as parameters) characterizing the information unit must be estimated, the probability of detection and the probability of false alarm related to a decision must be calculated. In the paper, a great deal of attention is paid to decision whereas estimation is set apart not because estimation is little important, but because estimation would require another research stream had it to be addressed to the appropriate level of exhaustivity.

Many tasks in data management are decision problems, examples are the classification of images with respect to predefined patterns, the categorization of webpages to topics, contextual advertising (i.e., the decision whether an ad has to displayed in a search engine result page), the retrieval and ranking of webpages (i.e., the decision as to whether a webpage has to put at rank r of a search engine result page), probabilistic databases (i.e., the decision about the correct value of an attribute and then the computation of the associated probability).

Our illustration of decision theory is necessarily brief and confined to its simplest aspects and examples. The illustration is also organized in such a way as to bring out most clearly the parallels between classical probability-based decision and quantum probability-based decision. The examples are chosen from elementary information retrieval or machine learning theory and perhaps provide a basis for comparison with the quantum case.

A certain information unit (e.g., a webpage or an store item) is observed in such a way as to obtain numbers (e.g., the PageRank or the number of positive reviews) on the basis of which a decision has to be made about its state. The numbers observed are, for example, the frequency of a fea-

ture in the information unit, the simplest example being the frequency of a keyword in a webpage used for calculating search engine statistical ranking functions. For the sake of clarity, we use the binary frequency and the feature presence/absence case in the paper. The state might be, for example, the relevance of the webpage to the search engine user’s interests or the customer’s willingness to buy the store item. The use of the term “state” is not coincidental because the numbers are observed depending upon the density matrix, which is indeed the mathematical notion implementing the state of a system. Thus, quantum probability ascribes the decision about the state of an information unit to testing the hypothesis that the density matrix has generated the observed numbers.

Consider the hypothesis that the state of the system is the density matrix ρ_1 and the alternative hypothesis that the state of the system is the density matrix ρ_0 . The two hypotheses can be labeled H_1 and H_0 , respectively. In data management, hypothesis H_0 asserts, for example, that a customer does not buy an item or that a webpage shall be irrelevant to the search engine user whereas hypothesis H_1 asserts that an item shall be bought by a customer or that a webpage shall be relevant to the user. Therefore, the probability that, say, a feature occurs in an item which shall not be bought by a customer or a keyword occurs in a webpage which shall be irrelevant to the search engine user depends on the state (i.e., the density matrix).

Statistical decision theory is a old topic and Neyman-Pearson’s lemma is by now one out of the most important results which provides a criterion for deciding upon hypotheses instead of the Bayesian approach. The lemma provides the rule to govern the decider’s behaviour and decide the true hypothesis without hoping to know whether it is true. Given an information unit and an hypothesis about the unit, such a rule calculates a specified number (e.g., a feature) and, if the number is greater than a threshold reject the hypothesis, otherwise, accept it. Such a rule tells nothing whether, say, the item shall be bought by the customer, but the lemma proves that, if the rule is followed, then, in the long run, the hypothesis shall be accepted at the highest probability of detection (or power) possible when the probability of false alarm (or size) [15] is not higher than a threshold. The set of the pairs given by size and power is the power curve which is also known as the Receiver Operating Characteristic (ROC) curve.

Neyman-Pearson’s lemma implies that the set of the observable numbers (e.g., features) can be partitioned into two distinct regions; one region includes all the numbers for which the hypothesis shall be accepted and is

termed acceptance region, the other region includes all the numbers for which the hypothesis shall be rejected and is termed rejection region. For example, if a keyword is observed from webpages and only presence/absence is observed, the set of the observable numbers is $\{0, 1\}$ and each region is one out of possible subsets, i.e., $\emptyset, \{0\}, \{1\}, \{0, 1\}$.

The paper reformulates Neyman-Pearson's lemma in terms of subspaces instead of subsets to utilize quantum probability. Therefore, the region of acceptance and the region of rejection must be defined in terms subspaces. In the following, we illustrate the algorithm for calculating the most efficient test in Hilbert spaces. The following result holds:

Theorem 3 *Let ρ_1, ρ_0 be the density matrices under H_1, H_0 , respectively. The region of acceptance at the highest power at every size is given by the projectors of the spectrum of*

$$\rho_1 - \lambda \rho_0 \quad \lambda > 0 \quad (8)$$

whose eigenvalues are positive.

Proof See [11]. ■

Definition 1 *An optimal projector is a projector which identifies the region of acceptance and the region of rejection according to Theorem 3.*

Definition 2 *We define the discriminant function as*

$$\text{tr}((\rho_1 - \lambda \rho_0)\mathbf{E}) \quad (9)$$

where \mathbf{E} is a projector. If the discriminant function is positive, the observed event represented by \mathbf{E} is placed in the region of acceptance.

Suppose that the density matrix that corresponds to H_1 is a mixed, classical probability distribution. The mixed case is the usual method for dealing with uncertainty in data management, even though more than one feature may exist or the feature may not be binary; however, the number of features or the number of values of a feature is not essential in the paper. Let μ_1 be such a mixed distribution and

$$\mu_1 = p_1 \mathbf{P}_1 + (1 - p_1) \mathbf{P}_0 = \begin{pmatrix} p_1 & 0 \\ 0 & 1 - p_1 \end{pmatrix} \quad (10)$$

where

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{P}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (11)$$

Similarly,

$$\mu_0 = p_0 \mathbf{P}_1 + (1 - p_0) \mathbf{P}_0 = \begin{pmatrix} p_0 & 0 \\ 0 & 1 - p_0 \end{pmatrix} \quad (12)$$

When \mathbf{P}_1 is observed, the power and the size are, respectively,

$$P_d = \text{tr}(\mu_1 \mathbf{P}_1) = p_1 \quad P_0 = \text{tr}(\mu_0 \mathbf{P}_1) = p_0 \quad (13)$$

In the classical case, $\mathbf{P}_0, \mathbf{P}_1$ represents the absence and the presence, respectively, of a feature. Hence, the possible acceptance or rejection regions are $\mathbf{0}, \mathbf{P}_0, \mathbf{P}_1, \mathbf{I}$ which correspond respectively to “never accept”, “accept when the feature does not occur”, “accept when the feature occurs” and “always accept”. Thus, the decision on, say, webpage classification, topic categorization, item suggestion, can be made upon the occurrence of one or more features because $\mathbf{P}_0, \mathbf{P}_1$ represent “physical” events. Furthermore, the discriminant function in the mixed case is

$$\text{tr}((\mu_1 - \lambda \mu_0) \mathbf{E}) \quad \mathbf{E} \in \{\mathbf{0}, \mathbf{P}_0, \mathbf{P}_1, \mathbf{I}\} \quad (14)$$

The power curve can be built as follows. Suppose, as an example, that a keyword describes webpage content and that that webpage either includes ($\mathbf{P}_1 = |1\rangle\langle 1|$) or does not include ($\mathbf{P}_0 = |0\rangle\langle 0|$) the keyword. $P_0 = 0$ only if $P_d = 0$ and this point corresponds to the event represented by $\mathbf{0}$. Let p_1, p_0 be the probability that the keyword occurs in a relevant webpage or in a non-relevant webpages, respectively. When the keyword is the unique observed feature, the webpage is presented to the user if $p_1 > \lambda p_0$ and the keyword occurs, or $p_1 < \lambda p_0 + (1 - \lambda)$ and the keyword does not occur. The power curve includes the points (p_0, p_1) and $(1 - p_0, 1 - p_1)$.

The key point is that a mixture is not the unique way to implement the probability distributions. As we illustrate in Section 2, the superposed vectors

$$|\varphi_1\rangle = \begin{pmatrix} \sqrt{p_1} \\ \sqrt{1 - p_1} \end{pmatrix} \quad |\varphi_0\rangle = \begin{pmatrix} \sqrt{p_0} \\ \sqrt{1 - p_0} \end{pmatrix} \quad (15)$$

yield the pure densities

$$\rho_1 = \begin{pmatrix} p_1 & \sqrt{p_1(1 - p_1)} \\ \sqrt{p_1(1 - p_1)} & 1 - p_1 \end{pmatrix} = |\varphi_1\rangle\langle\varphi_1| \quad (16)$$

$$\rho_0 = \begin{pmatrix} p_0 & \sqrt{p_0(1-p_0)} \\ \sqrt{p_0(1-p_0)} & 1-p_0 \end{pmatrix} = |\varphi_0\rangle\langle\varphi_0| \quad (17)$$

which replace the mixed densities. Theorem 3 instructs us to define the optimal projectors as those of the spectrum of (8) whose eigenvalues are positive, the spectrum being

$$\eta_0 \mathbf{Q}_0 + \eta_1 \mathbf{Q}_1 \quad \mathbf{Q}_0 = |\eta_0\rangle\langle\eta_0| \quad \mathbf{Q}_1 = |\eta_1\rangle\langle\eta_1| \quad (18)$$

where the η 's are eigenvalues,

$$\eta_0 = -R + \frac{1}{2}(1 - \lambda) < 0 \quad \eta_1 = +R + \frac{1}{2}(1 - \lambda) > 0 \quad (19)$$

and

$$R = \sqrt{\frac{1}{4}(1 - \lambda) + \lambda(1 - |X|^2)} \quad (20)$$

$$|X|^2 = \sqrt{p_0}\sqrt{p_1} + \sqrt{1-p_0}\sqrt{1-p_1} \quad (21)$$

$$= |\langle\varphi_0|\varphi_1\rangle|^2 \quad (22)$$

(see [11]). $|X|^2$ is the distance between densities defined in [24]; $|X|^2$ is the squared cosine of the angle between the subspaces corresponding to the density vectors. The justification of viewing $|X|^2$ as a distance comes from the fact that “the angle in a Hilbert space is the only measure between subspaces, up to a constant factor, which is invariant under all unitary transformations, that is, under all possible time evolutions.” [24]

Definition 3 $\mathbf{Q}_0, \mathbf{Q}_1$ are the optimal projectors in the pure case and $\mathbf{P}_0, \mathbf{P}_1$ are the optimal projectors in the mixed case.

The probability of detection (i.e., the power) Q_d and the probability of false alarm (i.e., the size) Q_0 in the pure case are defined as follows:

$$Q_d = \frac{\eta_1 + \lambda(1 - |X|^2)}{2R} \quad (23)$$

$$Q_0 = \frac{\eta_1 - (1 - |X|^2)}{2R} \quad (24)$$

Finally, Q_d can be defined as function of Q_0 :

$$Q_d = \begin{cases} \left(\sqrt{Q_0}\sqrt{|X|^2} + \sqrt{1-Q_0}\sqrt{1-|X|^2} \right)^2 & 0 \leq Q_0 \leq |X|^2 \\ 1 & |X|^2 < Q_0 \leq 1 \end{cases} \quad (25)$$

so that the power curve is obtained (see [11]).

Expressions (16) and (17) have no counterpart in classical probability and are among the essential points of the paper because they allow us to improve ranking yet using the same amount of evidence as the evidence used in the classical probability distribution (10) and (12).

At this point, there are three main issues:

- the numerical difference between the classical and the quantum probabilities of detection at every given probability of false alarm,
- the interpretations of $\mathbf{P}_0, \mathbf{P}_1$ and $\mathbf{Q}_0, \mathbf{Q}_1$ and whether the interpretations can be tied together,
- how the optimal projectors $\mathbf{Q}_0, \mathbf{Q}_1$ in the pure case can be used for ranking information units in a data management system.

The issues are addressed in Section 4, 5 and 6, respectively.

4 Optimal Projectors in the Quantum Space

The following lemma shows that the power of the decision rule in quantum probability is greater than, or equal to, the power of the decision rule in classical probability with the same amount of information available from the training set to estimate p_0, p_1 .

Lemma 1 *$Q_d \geq P_d$ at every given false alarm probability.*

Proof The equality holds only if $\mathbf{P}_i = \mathbf{Q}_i, i = 0, 1$:

$$\text{tr}(\rho_1 \mathbf{P}_1) = \begin{pmatrix} 1 & 0 \end{pmatrix} \rho_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = p_1 = \text{tr}(\mu_1 \mathbf{P}_1)$$

$$\text{tr}(\rho_0 \mathbf{P}_1) = \begin{pmatrix} 1 & 0 \end{pmatrix} \rho_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = p_0 = \text{tr}(\mu_0 \mathbf{P}_1)$$

Let $x \equiv P_0$ be a certain false alarm probability and let $Q_d(x), P_d(x)$ be the real, continuous functions yielding the detection probabilities at x . Q_d admits the first and the second derivatives in the range $[0, 1]$. In particular, $Q_d'' < 0$

in $[0, 1]$. P_d is a continuous function. Consider the polynomial $L_0(x)$ of order 1 passing through the points $(0, 1 - |X|^2)$ and (p_0, p_1) at which L_0 intersects Q_d . Then, the Lagrange interpolation theorem can be used so that

$$Q_d(x) - L_0(x) = Q_d''(c)x(x - p_0)/2$$

the latter being non negative because $Q_d'' < 0$ and $0 \leq x \leq p_0$. The number $c \in [0, p_0]$ exists due to the Rolle theorem. As $L_0(x) \geq P_d(x), x \in [0, p_0]$, hence, $Q_d(x) \geq P_d(x), x \in [0, p_0]$. Similarly, consider the polynomials $L_1(x)$ and $L_2(x)$ of order 1 passing through the points (p_0, p_1) , $(1 - p_0, 1 - p_1)$ and $(1 - p_0, 1 - p_1)$, $(1, 1)$ at which L_1 and L_2 intersect Q_d , respectively. Then, the Lagrange interpolation theorem can again be used so that

$$Q_d(x) - L_1(x) = Q_d''(c)(x - p_0)(x - 1 + p_0)/2$$

$$Q_d(x) - L_2(x) = Q_d''(c)(x - 1 + p_0)(x - 1)/2$$

Then, $Q_d(x) \geq P_d(x)$ for all $x \in [0, 1]$. ■

The power Q_d can plotted against the size Q_0 , thus producing the power curve of the classical decision rule and the power curve of the quantum decision. A graphical representation is provided in Figure 2.

Example 1 *Suppose that ten information units have been used for training a data management system. Each unit has been indexed using one binary feature and has been marked as useful (1) or useless (0). The training set is summarized by Table 1:*

| <i>unit</i> | u_1 | u_2 | u_3 | u_4 | u_5 | u_6 | u_7 | u_8 | u_9 | u_{10} |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| <i>feature</i> | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| <i>use</i> | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

Table 1: The training set of Example 1

Therefore,

$$p_1 = \frac{3}{5} \quad p_0 = \frac{1}{5} \quad |X|^2 = 0.91 \quad d^2 = 0.09 .$$

The computation of P_d, P_0 follows from (13) and the computation of Q_d, Q_0 follows from (23). When $P_0 = Q_0$, we have that $Q_d = P_d$.

5 Interpretation of the Optimal Projectors in Data Management

In this section, some interpretations of the optimal projectors representing the region of acceptance are provided. The optimal projectors $\mathbf{Q}_0, \mathbf{Q}_1$ in the pure case have a more difficult interpretation than $\mathbf{P}_0, \mathbf{P}_1$ because the latter represent “physical” observations (e.g., a customer review does exist or does not) whereas $\mathbf{Q}_0, \mathbf{Q}_1$ cannot be expressed in terms of $\mathbf{P}_0, \mathbf{P}_1$ and we cannot explain the \mathbf{Q}_i ’s by saying that, for example, they represent the presence and/or the absence of a feature. In quantum theory, the impossibility of expressing a projectors as functions of other projectors is termed incompatibility which is expressed mathematically as $\mathbf{Q}_i \mathbf{P}_j \neq \mathbf{P}_j \mathbf{Q}_i, \mathbf{P}_i \neq \mathbf{Q}_i$.

The interpretation of the optimal projectors reflects on the interpretation of what means that they are “observed” in an information unit; for example, if the information unit is a commercial item suggested to a customer, what does the “observation” of \mathbf{Q}_1 mean? What should we observe from an information unit so that the observation outcome corresponds to the projector? The question is not a futile because the answer(s) would effect the algorithms (e.g., automatic indexing) used for representing the informative content of the unit. Specifically, the interpretation of $\mathbf{Q}_0, \mathbf{Q}_1$ provides what the retrieval algorithm must do when a feature is observed. Either the interpretation of an optimal projector is implemented at indexing time or at query time, an internal memory representation in terms of data structures is necessary for automatic processing and the representation needs the observation of physical properties which are then converted into numbers. In the mixed case, the answer is quite straightforward because the optimal projectors correspond to the feature occurrence and separate the units indexed by the feature from those not indexed. In the pure case, the answer is not straightforward at all. If $\mathbf{P}_0, \mathbf{P}_1$ represent feature occurrence, the \mathbf{Q} ’s cannot be a feature occurrence, but they are something new which cannot be described in conventional way.

Geometrically, each vector is a superposition of other two independent vectors. Figure 3 depicts the way the vectors and the spanned subspaces (i.e., projectors) interact and shows that $|\eta_1\rangle, |\eta_0\rangle$ are placed symmetrically “around” the density vectors and the probability of error is minimized [11]. The observation of a binary feature places the observer upon either $|0\rangle$ or $|1\rangle$ and there is no way to move upon $|\eta_0\rangle$ or $|\eta_1\rangle$.

Probabilistically, the optimal projectors and the density vectors are related as follows:

$$|\varphi_0\rangle = \sqrt{1-Q_0}|\eta_0\rangle + \sqrt{Q_0}|\eta_1\rangle \quad (26)$$

$$|\varphi_1\rangle = \sqrt{1-Q_d}|\eta_0\rangle + \sqrt{Q_d}|\eta_1\rangle \quad (27)$$

where

$$Q_d = |\langle\eta_1|0\rangle\langle 0|\varphi_1\rangle + \langle\eta_1|1\rangle\langle 1|\varphi_1\rangle|^2 \quad (28)$$

$$Q_0 = |\langle\eta_1|0\rangle\langle 0|\varphi_0\rangle + \langle\eta_1|1\rangle\langle 1|\varphi_0\rangle|^2 \quad (29)$$

Logically, the projectors are assertions, thus a parallel can be established with assertions and subsets – an assertion defines the elements of the universe (e.g., an event space) which belong to a subset. The basic difference between subspaces and subsets is that the vectors belong to a subspace if and only if they are spanned by a basis of the subspace. A containment relationship can be established between subspaces such that if a subspace (e.g., a line) includes a point, then every subspace (e.g., a plane) containing the line includes the point too. The subspace spanned by the projector \mathbf{A} is termed as $L(\mathbf{A})$ and the containment relationship between $L(\mathbf{A})$ and $L(\mathbf{B})$ can be encoded as $\mathbf{AB} = \mathbf{B}$ such that for every vector $|x\rangle$, $\mathbf{AB}|x\rangle = \mathbf{B}|x\rangle$ [22, Ch. 5].

The paper considers the information units relevant, useful or interesting when they are included by the subspace $L(|\varphi_1\rangle)$ spanned by $|\varphi_1\rangle$. Suppose that a subspace $L(\mathbf{A})$ is given and that $|\langle x-y|x-y\rangle|^2$ is the metric defined on the whole space to which $L(\mathbf{A})$ belongs.

Proposition 1 $|y^*\rangle = \arg_{|y\rangle \in L(\mathbf{A})} \min |\langle x-y|x-y\rangle|^2 = \mathbf{A}|x\rangle$

Proof See [14, 22]. ■

As $|\langle x-y^*|x-y^*\rangle|^2 = \text{tr}(|x-y^*\rangle\langle x-y^*|) = 1 - |\langle x|y^*\rangle|^2$, $|y^*\rangle$ maximizes the probability

$$\text{tr}(|x\rangle\langle y^*|) = |\langle x|y^*\rangle|^2 \quad (30)$$

Note that when $|x\rangle$ is in $L(|\varphi_0\rangle)$, (30) is the distance between the $L(|\varphi_0\rangle)$'s defined in [24].

The result establishes a connection between the geometric, probabilistic and logical interpretation of the projectors even though they seems different. In the next section, these interpretations are tied together, thus allowing us to look for a criterion for assigning an information unit to the best region as explained.

6 Implementation of the Optimal Ranking

The problem is to decide whether an information unit represented by a binary feature³ is considered relevant, useful, interesting, etc.. An algorithm implementing such a decision rule should perform as follows. It reads the feature occurrence symbol (i.e., either 0 or 1); check whether the feature is included by the region of acceptance. If the feature is not included, the hypothesis of relevance, usefulness, interest, etc. is rejected.

Another view of the preceding decision rule is the ranking of the information units. When ranking information units, the system returns the units whose features lead to the highest probability of detection, then those whose features lead to the second highest probability of detection, and so on. When a binary feature is considered, the ranking ends up to placing the units whose features lead to the highest probability of detection whereas the other units are not retrieved.

As we point out in Section 5, the observation of features corresponding to $\mathbf{P}_1, \mathbf{P}_0$ cannot give any information about the observation of the events corresponding to $\mathbf{Q}_1, \mathbf{Q}_0$ due to the incompatibility between these pairs of events. Thus, we cannot design an algorithm implementing the decision rule so that the observation of a feature can be translated into the observation of the events corresponding to $\mathbf{Q}_0, \mathbf{Q}_1$.

A possible approach can be based on the probabilistic interpretation of the optimal projectors. According to such an approach, the probability that the event corresponding to an optimal projector occurs provided that a feature occurs can provide a measure of the degree to which the event had occurred if it could have been observed. When the subspaces represent these events, the probability that the event corresponding to an optimal projector is observed if the state is described by ρ_1 is P_d . Such an approach is partially satisfactory because $P_d \neq Q_d$.

As an alternative approach, consider the geometric interpretation depicted in Figure 3. Note that the asymmetry of $|0\rangle, |1\rangle$ with respect to the densities causes the suboptimality of $\mathbf{P}_1 = |1\rangle\langle 1|, \mathbf{P}_0 = |0\rangle\langle 0|$. Indeed, if $|0\rangle, |1\rangle$ coincided with $|\eta_0\rangle, |\eta_1\rangle$, the power and the size would be the same in both cases.

We propose a method which reaches the optimality without neither resorting to probabilistic approximations nor undergoing high computational

³We recall that the binary case has been introduced for the sake of clarity.

costs.

The density vectors $|\varphi_1\rangle, |\varphi_0\rangle$ are a superposition of both $|0\rangle, |1\rangle$ and $|\eta_0\rangle, |\eta_1\rangle$ which are two different bases and induce different coordinates. When $|0\rangle, |1\rangle$ is the basis, the coordinates of $|\varphi_i\rangle$ are $\sqrt{p_i}, \sqrt{1-p_i}$. When $|\eta_0\rangle, |\eta_1\rangle$ is the basis, the coordinates of $|\varphi_i\rangle$ are $x_{00}, x_{01}, x_{10}, x_{11}$ such that

$$|\varphi_0\rangle = x_{00}|\eta_0\rangle + x_{01}|\eta_1\rangle \quad |\varphi_1\rangle = x_{10}|\eta_0\rangle + x_{11}|\eta_1\rangle \quad (31)$$

and

$$x_{00}^2 = \frac{|X|^2}{(1-\eta_1)^2 + |X|^2} \quad x_{01}^2 = \frac{(1-\eta_1)^2}{(1-\eta_1)^2 + |X|^2} \quad (32)$$

$$x_{10}^2 = \frac{|X|^2}{(1+\eta_1)^2 + |X|^2} \quad x_{11}^2 = \frac{(1+\eta_1)^2}{(1+\eta_1)^2 + |X|^2} \quad (33)$$

As $\lambda = 1$ is often assumed when ranking information units, the coordinates have a quite simple and intuitive meaning provided by the following expressions:

$$x_{00}^2 = \frac{1+d^2}{2} \quad x_{01}^2 = \frac{1-d^2}{2} \quad (34)$$

$$x_{10}^2 = \frac{1-d^2}{2} \quad x_{11}^2 = \frac{1+d^2}{2} \quad (35)$$

where $1-d^2 = |X|^2$.

As the asymmetry of $|0\rangle, |1\rangle$ with respect the density vectors is due to p_1, p_0 which summarize the statistics observed from the training set, we leverage them for improving the ranking. In particular, in the paper we show that changing the estimation of p_1, p_0 is sufficient to reach the optimality.

Then, we wonder how we should define the density vectors or matrices so that Q_d, Q_0 were obtained instead of P_d, P_0 . The basis vectors (i.e., $|0\rangle, |1\rangle$ or $|\eta_0\rangle, |\eta_1\rangle$) are rotated, thus changing the coordinates.

Therefore, we define the density vectors $|\varphi_0\rangle, |\varphi_1\rangle$ in $|0\rangle, |1\rangle$ according to (31) in such a way that if a feature is observed under H_1 , the probability of detection is Q_d and, if a feature is observed under H_0 , the probability of false alarm is Q_0 . The simple solution is defining the new density vectors as follows:

$$|\varphi'_0\rangle = x_{00}|0\rangle + x_{01}|1\rangle \quad |\varphi'_1\rangle = x_{10}|0\rangle + x_{11}|1\rangle \quad (36)$$

thus obtaining

$$Q'_0 = \text{tr}(|\varphi'_0\rangle\langle\varphi'_0|\mathbf{P}_1) = Q_0 \quad Q'_d = \text{tr}(|\varphi'_1\rangle\langle\varphi'_1|\mathbf{P}_1) = Q_d \quad (37)$$

| First Eigenvalue | Second Eigenvalue | |
|---------------------|------------------------------|------------------------------|
| | $1 - p_1 < \lambda(1 - p_0)$ | $1 - p_1 > \lambda(1 - p_0)$ |
| $p_1 < \lambda p_0$ | $\mathbf{0}$ | \mathbf{P}_0 |
| $p_1 > \lambda p_0$ | \mathbf{P}_1 | \mathbf{I} |

Table 2: The regions of acceptance corresponding to the sign of the eigenvalues of the spectrum of the discriminant function in the mixed case. The equality case is addressed in [11].

| First Eigenvalue | Second Eigenvalue | |
|----------------------------------|--|--|
| | $1 - x_{11} ^2 < \lambda(1 - x_{01} ^2)$ | $1 - x_{11} ^2 > \lambda(1 - x_{01} ^2)$ |
| $ x_{11} ^2 < \lambda x_{01} ^2$ | $\mathbf{0}$ | \mathbf{Q}_0 |
| $ x_{11} ^2 > \lambda x_{01} ^2$ | \mathbf{Q}_1 | \mathbf{I} |

Table 3: The regions of acceptance corresponding to the sign of the eigenvalues of the spectrum of the discriminant function in the pure case. The equality case is addressed in [11].

At first sight, the increase of the probability of detection is due to the higher probability values assigned to the region of acceptance in the pure case than those assigned to the region of acceptance in the mixed case, and not to a different ranking. In the following, we show that the superiority of the discriminant function in the pure case is due to the different ranking induced by a different partition of the event space into region of acceptance and region of rejection.

We state the problem as follows. Are there p_0, p_1, λ such that the region of acceptance in the pure case differs from that in the mixed case? Consider Theorem 3 to answer the question. The region of acceptance in the mixed case is defined through Table 2 whereas the region of acceptance in the pure case is defined through Table 3. Furthermore, the discriminant function derived from (36) is

$$\text{tr}((\sigma_1 - \lambda\sigma_0)\mathbf{E}) \quad \mathbf{E} \in \{\mathbf{0}, \mathbf{Q}_0, \mathbf{Q}_1, \mathbf{I}\} \quad (38)$$

where

$$\sigma_i = \begin{pmatrix} |x_{i1}|^2 & \sqrt{|x_{i1}|^2(1 - |x_{i1}|^2)} \\ \sqrt{|x_{i1}|^2(1 - |x_{i1}|^2)} & 1 - |x_{i1}|^2 \end{pmatrix} \quad i = 0, 1 \quad (39)$$

Suppose that $p_0 = 1, p_1 = \frac{7}{10}, \lambda = \frac{1}{2}$. Thus, $X^2 = \frac{7}{10}, R \cong 0.21, x_{11}^2 \cong 0.81, x_{01}^2 \cong 0.11$ and the region of acceptance in the mixed case is represented by \mathbf{I} , whereas the region of acceptance in the pure case is represented by \mathbf{P}_1 . The counter-example just mentioned proves the following

Corollary 1 *The discriminant function (38) ranks information units in a different way from the discriminant function (14) because an alternative ranking is computed.*

In (14) the densities that are considered are those associated to a mixed state, while (38) in the densities are the one associated to a pure state. So the equations look like the same, what differs is the type of densities that are used in the two cases. We have shown that the improvement of ranking measured in terms of probability of detection given a probability of false alarm is due to the ranking induced by Q'_0, Q'_d .

Hence, we state the problem of finding $\mathbf{Q}_0, \mathbf{Q}_1$ into the problem of defining the coordinates of the representation of the density vectors in $|0\rangle, |1\rangle$. The problem of defining the new coordinates for the density vectors might be viewed as a problem of feature weighting. In such a context, the traditional estimations of the probability of feature occurrence (i.e., p_1, p_0) under two different hypothesis are replaced by the x_{ij} 's. Feature re-weighting is explored in IR whose state-of-the-art is given by the BM25 weighting scheme surveyed, for example, in [21]. The main drawback of the weighting schemes like BM25 is the parameter tuning necessary for optimizing the effectiveness with a given database or query, thus making the understanding of how and why a scheme is more effective than others rather problematic.

In contrast, the paper illustrate the decision rule in such a way that if the decision rule is followed, then H_1 shall be accepted when it is true at a higher probability of detection than P_d when the probability of false alarm is not more than a given threshold. The formulation of the decision rule provided in this section allows us to design an efficient algorithm for indexing and retrieving (or classifying) information units. The algorithm is just an instance of those employed currently in IR (see [7] for example) where the maximum likelihood or Bayesian estimations of p_0, p_1 are replaced by (34) and (35).

7 Related Work

The foundations of quantum mechanics and theory has been illustrated in plenty of books such as [9] and [12]. Quantum probability, for example, has been introduced in [17]. In particular, the interference term is addressed in [1]. The view of probability illustrated in Section 2 is based on [19]. The utilization of quantum theory in computation, information processing and communication is described in [16]. Recently, investigations have started in other research areas, for example, in IR [5].

The paper is inspired by Helstrom’s book [11] which provides the foundations and the main results in quantum detection; an example of the exploitation of the results in quantum detection is reported [6] within communication theory. This paper links to [22] as far it concerns density matrices and projectors; however, the paper develops quantum detection for data management.

This paper departs from the Probability Ranking Principle (PR) proposed in the context of classical probability; we propose quantum probability to improve ranking in a principled way. In Information Retrieval, the Probability Ranking Principle (PRP) states that “If a reference retrieval system’s response to each request is a ranking of the documents in the collection in order of decreasing probability of relevance to the user who submitted the request, where the probabilities are estimated as accurately as possible on the basis of whatever data have been made available to the system for this purpose, the overall effectiveness of the system to its user will be the best that is obtainable on the basis of those data.” [20]. However, some assumptions undermine the general applicability of the PRP. We state a similar principle yet replace classical probability, which is implied in [20], with quantum probability – parameter estimation data are kept the same, but we instead use subspaces to define alternative regions of acceptance and rejection.

To our knowledge, the use of quantum probability for ranking information units has not yet been addressed in the same way of this paper although a few papers that are somehow comparable can be found. Perhaps, the closest paper is [25]. That paper proposes to rank documents by quantum probability and suggests that interference (which must be estimated) might model dependencies in relevance judgements such that documents ranked until position $n-1$ interfere with the degree of relevance of the document ranked at position n . This means, the optimal order of documents under the PRP differs from that of the Quantum PRP. Note that they empirically show that quantum probability is more effective than classical one in specific rankings

tasks.

In contrast, in this paper, we do not need to address interference because quantum probability can be estimated using the same data used to estimate classical probability. We rather show that not only ranking by quantum probability provides a different optimal ranking, it is also more effective than classical probability. With this regard, the effectiveness of quantum probability measured in [25] stems from the estimation of classical probability and that of interference. But, the regions of acceptance and rejection are still based on sets. It follows that the optimality of the Quantum PRP strongly depends on the optimality of the PRP and on the interference estimation effectiveness. In this paper, on the contrary, ranking optimality only depends on the region of acceptance defined upon subspaces.

Another paper somewhat related to ours is [18]. The authors discuss how to employ quantum formalisms for encompassing various Information Retrieval tasks within a single framework. From an experimental point of view, what that paper demonstrates is that ranking functions based on quantum formalism are computationally feasible. The best experimental results of rankings driven by quantum formalism are comparable to BM25, that is, to PRP, thus limiting the contribution within a classical probability framework.

Probabilistic databases systems manage imprecise data and provide tools for structured complex queries. A survey is provided in [8]. Beside scalability and query plan execution, these systems do probabilistic inference which may be defined upon classical or quantum probability and they concentrate on top- k query answering where the tuples are assigned a probability distribution. The results of this paper may be applied to probabilistic databases systems too.

8 Future Developments and Conclusions

The main result of the paper is the demonstration that quantum probability can be incorporated into a data management system for ranking information units. As ranking by quantum probability is more effective than ranking by classical probability when it has been used in other domains, it is our belief that an analogous improvement can be achieved within data management.

The future developments are threefold. First, we will work on the interpretation of the optimal projectors in the pure case because the detection of them in an information unit may open further insights. Second, feature

classical correlation and quantum entanglement will be investigated. Third, evaluation is crucial to understand whether the results of the paper can be confirmed by the experiments.

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A Dirac Notation

A complex vector x is represented as $|x\rangle$ and is called “ket”. The conjugate transpose of x is represented as $\langle x|$ and is called “bra” (therefore, the Dirac notation is called the bra(c)ket notation).

The inner product between x and y is represented as $\langle x|y\rangle$, which is a complex number. The inner product between x and itself is $\langle x|x\rangle$.

The outer product (or dyad) is $|x\rangle\langle y|$. A special case of dyad is $|x\rangle\langle x|$ which is the projector made onto x .

If A is a matrix (or an operator), then $A|x\rangle$ is the vector resulting from the linear transformation represented by A . $|x\rangle\langle x|$ is also an operator because it is a projector.

The real number $|\langle x|y\rangle|^2$ is the squared inner product between x and y .

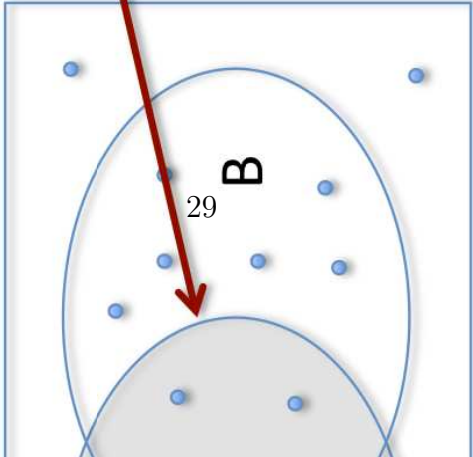
Moreover, $\langle x|\mathbf{A}|y\rangle = \text{tr}(\langle x|\mathbf{A}|y\rangle)$ and the properties of trace allow us to write $\text{tr}(\langle x|\mathbf{A}|y\rangle) = \text{tr}(\mathbf{A}\langle x|y\rangle)$. In general, if \mathbf{A}, \mathbf{B} are trace-1 Hermitian operators, and \mathbf{B} is a projector, $0 \leq \text{tr}(\mathbf{A}\mathbf{B}) \leq 1$ is the probability that the event represented by \mathbf{B} occurs given a density operator \mathbf{A} .

The Dirac notation allow us to write complex expressions in an elegant way, for example,

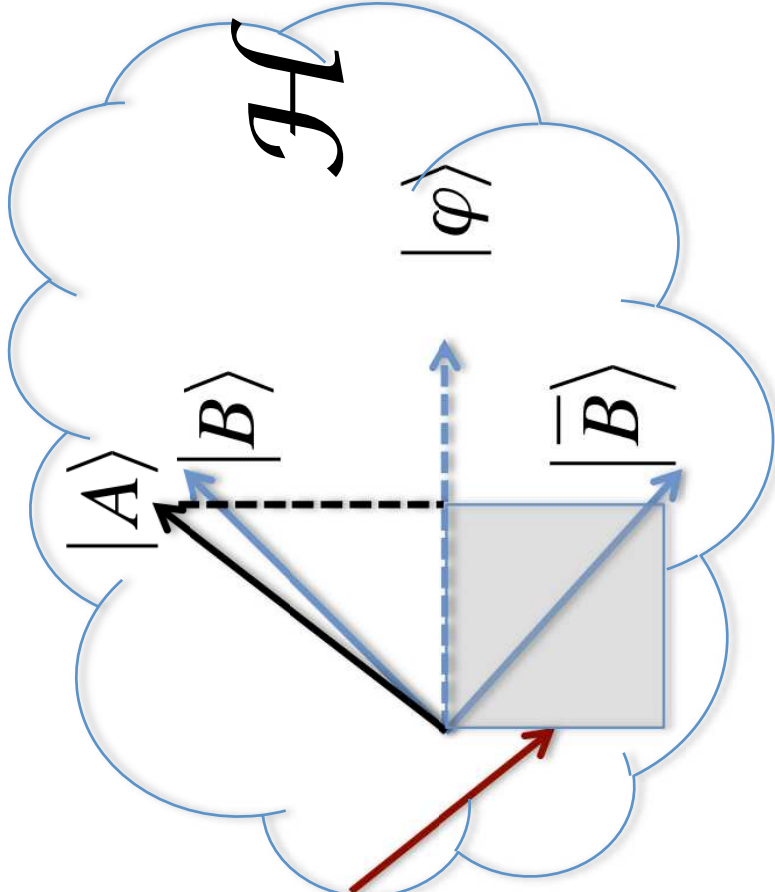
$$\begin{aligned} \langle x|y\rangle\langle y|z\rangle\langle z|x\rangle &= \text{tr}(\langle x|y\rangle\langle y|z\rangle\langle z|x\rangle) \\ &= \text{tr}(|x\rangle\langle x||y\rangle\langle y||z\rangle\langle z|) \\ &= \text{tr}(\langle z|x\rangle\langle x|y\rangle\langle y|z\rangle) \end{aligned}$$

When \mathbf{A} is mixed and \mathbf{E} is a projector,

$$\begin{aligned} \text{tr}(\mathbf{A}\mathbf{E}) &= \text{tr}\left(\left(\sum_{i=0}^{r-1} \alpha_i \mathbf{E}_i\right) \mathbf{E}\right) \\ &= \text{tr}\left(\sum_{i=0}^{r-1} \alpha_i \mathbf{E}_i \mathbf{E}\right) \\ &= \sum_{i=0}^{r-1} \alpha_i \text{tr}(\mathbf{E}_i \mathbf{E}) \end{aligned}$$



$$P(A) = \frac{6}{15}$$



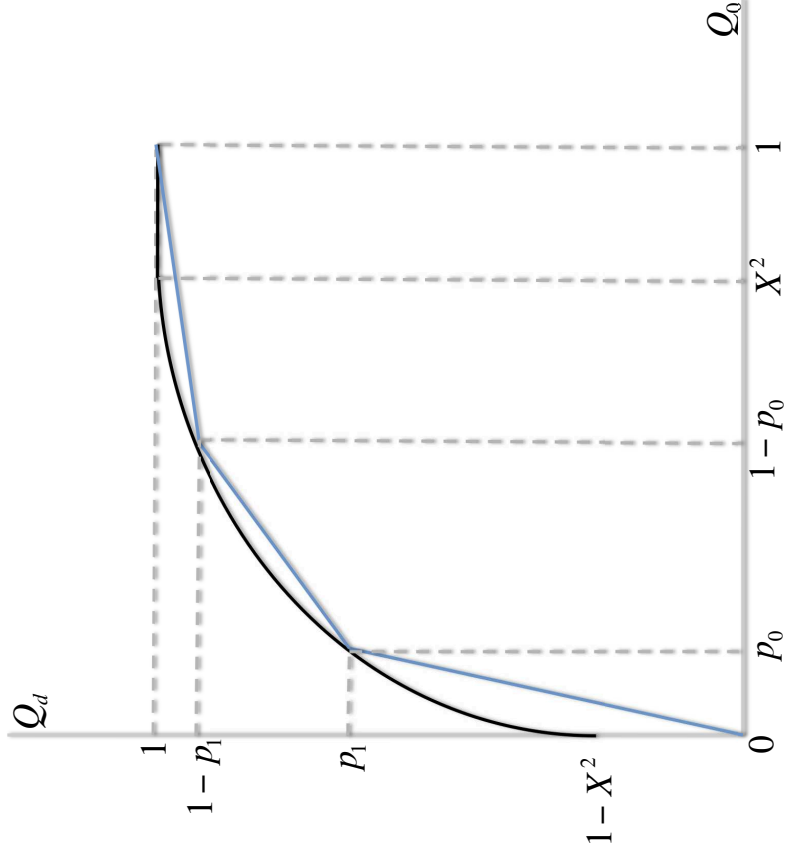


Figure 2: A graphical representation of the ROC curves. Q_d is the curve above the polygonal curve depicting P_d . The classical probability ROC curve intercepts the quantum probability ROC curve at (p_0, p_1) , $(1-p_0, 1-p_1)$ and $(1-p_0, 1-p_1)$, $(1, 1)$ where $\mathbf{P}_1, \mathbf{P}_0$ are observed. $Q_d = 1$ for $Q_0 > |X|^2$ and $Q_d \geq P_d$ for all Q_0 's.

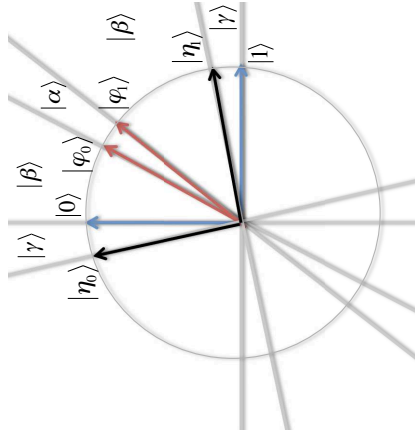


Figure 3: The geometric view of the canonical vectors in the mixed case, the optimal vectors in the pure case and the vectors representing the densities. The Greek letters α, β, γ indicate the angles between the vectors.